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# Anomalous muon magnetic moment in a left-right symmetric composite model

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## Abstract

We calculate the anomalous magnetic moment for muons at one loop level arising from left right symmetric excited leptons which are excited states of known standard model leptons. Such excited states arise in compositeness theories where the known leptons are assumed to be made of more fundamental particles. In this work, we assume that the excited leptons possess a left right symmetry also. We show that at the one loop level, the QED contribution to the muon anomalous magnetic moment from these excited leptons comes from only one Feynman diagram, which turns out to be a natural analog of the only diagram contributing to anomalous muon magnetic moment in the Standard Model.

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## I. INTRODUCTION

In 1948 Schwinger [1], Feynman [2] and Tomonaga [3] showed that a charged lepton interacting with external electromagnetic field, would give rise to a magnetic moment which, in units of Bohr magneton, is given by

$$\mu_l = g_l \left( \frac{e_l}{2m_l c} \right) \hbar \frac{\sigma}{2}$$

where  $g_l$  is the gyromagnetic factor of the lepton. The gyromagnetic factor represents the relative strength of the intrinsic magnetic dipole moment to the strength of the spin-orbit coupling for the lepton. The Dirac equation predicts  $g_l = 2$ . The value of  $g_l$  would shift if contributions from loop diagrams involving QED, weak and strong interactions are taken into account. This shift is known as anomalous magnetic moment( $a_l$ ) defined as  $(g_l - 2)/2$ .

Several experiments have been conducted to precisely measure the anomalous magnetic moment for electrons and muons. For muons the very precise experimental value [4] of

$$a_\mu^{\text{exp}} = 11659208.9(6.3) \cdot 10^{10}$$

is due to the experiment E821 carried out at the Brookhaven laboratories [5, 6]. The experimental results are at about  $3\sigma$  away from the Standard Model (SM) theoretical predictions [7]. New experiments are underway at Fermilab [6, 8] and at J-PARC [9] to confirm the above value and reduce the experimental uncertainty. In order to reach an accuracy comparable to experimental results theoretical calculations of  $a_\mu$  have been reviewed and revisited [10]. Thus considering both the experimental result and the SM calculation on equally firm footings, the  $3\sigma$  discrepancy clearly hints at new physics beyond SM. Several new models aiming to address various shortcomings of SM may be able to explain the discrepancy, at least in part. For example, many Beyond Standard Model theories which predict a new spectrum of fermions also contribute to the magnetic moment for known leptons at one loop level. At present in the absence of any direct evidence of beyond Standard Model particles, the  $g - 2$  experimental results serve to constrain the parameter space of such models.

Composite models predict new spectra of fermions which can contribute to the anomalous magnetic moment of the known leptons. In these models known SM fermions are not treated as point like but composite [11], thereby predicting a rich spectrum of excited fermions. The known fermions can be thought of as the ground state of these excited fermions. Excited fermions can contribute a large anomalous magnetic moment to ordinary SM leptons, as investigated in several works [12–14]. However, these works assume that excited leptons only obey left handed symmetry just like ordinary leptons.

The confirmation of small neutrino masses [15–18] in 1998 has strengthened the possibility that the spectrum of matter is after all symmetric between the two handedness states, with the maximal parity violating weak forces being a low energy effect. The Left-Right symmetric model [19, 20] through the see-saw mechanism [21–23] can explain the small neutrino masses [24]. In this model both left and right handed fermions are taken as doublets. If the excited lepton spectrum together with its ground state has both chiralities, that is, if there are both left and right handed leptons

along with their excited states [25], then the contribution to the anomalous magnetic moment of ordinary SM leptons will be different. The contribution to  $a_\mu$  from the spectrum of the left right symmetric composite model can be used to explain the experimental measurements and hence can be used to constrain its parameter space.

## II. LEFT-RIGHT SYMMETRIC EXCITED LEPTON MODEL

For the left-right symmetric composite model of [25] the magnetic transition between ordinary lepton and the excited lepton is given by

$$\mathcal{L}_{\text{trans}} = \frac{1}{2\Lambda} \bar{l}_L^* \sigma^{\mu\nu} \left[ g_s f_s \frac{\lambda^a}{2} G_{\mu\nu}^a + g_1 f_1 \frac{\tau}{2} \cdot \mathbf{W}_{\mu\nu}^L + g_2 f_2 \frac{\tau}{2} \cdot \mathbf{W}_{\mu\nu}^R + g'' f'' \frac{B-L}{2} B_{\mu\nu}^{B-L} \right] l_R + \text{H.c.} \quad (1)$$

Here  $W_{\mu\nu}^L$  and  $W_{\mu\nu}^R$  are the field strength tensors of  $SU(2)_L$  and  $SU(2)_R$  gauge fields respectively and  $B_{\mu\nu}^{B-L}$  is the field strength tensor of  $U(1)_{B-L}$ .  $g_1$ ,  $g_2$  and  $g''$  are the  $SU(2)_L$ , the  $SU(2)_R$  and the  $U(1)_{B-L}$  gauge couplings respectively. Consistent with the original left-right symmetry philosophy we assume  $g_1 = g_2$ .  $f_1$ ,  $f_2$  and  $f''$  are the new couplings that arise due to compositeness in the theory. The possible gauge mediated transitions between ordinary leptons and excited fermions of both chiralities arising from the above Lagrangian are shown in Fig. 1.

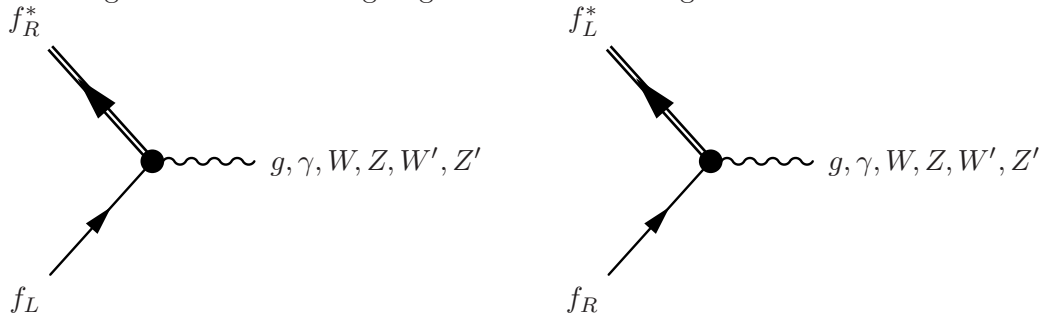


FIG. 1. Transitions between ordinary and left-right symmetric fermions via gauge boson emission.

When spinors couple to photons it gives rise to magnetic dipole moment. Part of the discrepancy between experimental and SM theoretical predictions of muon anomalous magnetic moment may be explained by the contribution of left right symmetric excited muons via photon emission. Even though in principle excited states with higher spin and various isospin values might exist and contribute to the anomalous muon magnetic moment, there is to date no compelling reason proposed for any fine cancellation between the contributions from the first excited state and other excited states. Hence, for a conservative order of magnitude estimation we will consider only the contribution of the lowest lying excited state of spin and isospin 1/2 at the one loop level, and obtain constraints on the couplings  $f_1$ ,  $f_2$ ,  $f''$  by comparing it with the discrepancy between theoretical SM and experimental results.

## III. MAGNETIC MOMENT FOR LEFT RIGHT SYMMETRIC MODEL

The contribution of excited muons to the anomalous magnetic moment of ordinary muons at one loop level occurs from the Feynman graphs shown in Table I. Table I also lists the corresponding

Feynman integrals. In the simplest proposals, the effective dipolar coupling for excited leptons to leading order in derivatives is introduced through a dipolar form factor for the excited leptons, given by  $\frac{\Lambda^4}{(q^2 - \Lambda^2)^2}$  [13, 14], where  $q^2$  is the virtual photon mass squared. We tabulate the contributions to the form factor,  $F_2(0)$ , for all the leading graphs in Tables II, III, IV, V.

However, except for graph 1 of Table I all other graphs have their corresponding mirror images. The contributions to matrix element from the Feynman graphs 2, 3 and 4 is exactly opposite to those from their mirror images 2', 3' and 4'. This is nothing but a manifestation of the gauge invariance of the underlying theory. It is interesting to note that the only non-zero SM contribution to anomalous muon magnetic moment at the one loop level comes from the SM analog of graph 1. In our left-right symmetric theory barring the form factor for excited lepton and the effective charge nothing changes in the expression for matrix element. Since in our left-right symmetric theory the only contribution to muon anomalous magnetic moment that survives comes from graph 1, we now proceed to provide some details of the calculation of its contribution to  $F_2(0)$ .

Employing the Feynman rules on graph 1 we get,

$$iM^\mu = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\alpha}}{k^2 + i\epsilon} \bar{u}(q_2)(e_{\text{eff}}k_\beta\sigma^{\nu\beta}) \frac{1}{(1 - \frac{k^2}{\Lambda^2})^2} \frac{i(q_2 + k + M)}{(q_2 + k)^2 - M^2 + i\epsilon} (-ie\gamma^\mu) \\ \frac{i(q_1 + k + M)}{(q_1 + k)^2 - M^2 + i\epsilon} (e_{\text{eff}}k_\beta\sigma^{\alpha\beta}) \frac{1}{(1 - \frac{k^2}{\Lambda^2})^2} u(q_1),$$

where  $\Lambda$  is the compositeness scale,  $m$  the mass of the ordinary lepton and  $M$  the mass of the excited lepton. The term  $e_{\text{eff}}$  above is the effective charge which turns out to be

$$e_{\text{eff}} = \frac{e}{\Lambda}(f_1 + f_2 + f''),$$

where  $f_1$ ,  $f_2$  and  $f''$  are the couplings present in the left-right symmetric theory of compositeness.

Using standard techniques for evaluating Feynman integrals, it can be shown that in the regime  $m \ll M \leq \Lambda$ , the contribution to the magnetic moment obeys the following equation.

$$F_2(0) = \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} (f_1 + f_2 + f'')^2 \int_0^1 dt \\ \left( \frac{\frac{t}{2} - \frac{t^2}{3}}{1 - t + tM^2\Lambda^{-2}} + \frac{3t^2(1-t)M^2\Lambda^{-2}}{1 - t + tM^2\Lambda^{-2}} - \frac{3t^3(1-t)M^4\Lambda^{-4}}{2(1 - t + tM^2\Lambda^{-2})^2} + \frac{t^4(1-t)M^6\Lambda^{-6}}{3(1 - t + tM^2\Lambda^{-2})^3} - \frac{11t(1-t)}{6} \right) \\ + \text{lower order.}$$

Above, the phrase ‘‘lower order’’ means that the values of the remaining summands are at least a factor of  $1/M$  lower than the given expression.

The above integral can be evaluated explicitly using standard methods to get

$$F_2(0) = (f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot 8Mm\Lambda^{-2} \left( \frac{1}{2}I_1 + \left( 3M^2\Lambda^{-2} - \frac{1}{3} \right) I_2 - 3M^2\Lambda^{-2}I_3 - \frac{3M^4\Lambda^{-4}}{2}I_4 \right. \\ \left. + \frac{3M^4\Lambda^{-4}}{2}I_5 + \frac{M^6\Lambda^{-6}}{3}I_6 - \frac{M^6\Lambda^{-6}}{3}I_7 - \frac{11}{36} \right),$$

where

$$I_1 = \frac{1}{(1 - M^2\Lambda^{-2})^2} (-\ln(M^2\Lambda^{-2}) - (1 - M^2\Lambda^{-2})),$$

$$I_2 = \frac{1}{(1 - M^2 \Lambda^{-2})^3} \left( -\ln(M^2 \Lambda^{-2}) - 2(1 - M^2 \Lambda^{-2}) + \frac{1 - M^4 \Lambda^{-4}}{2} \right),$$

$$I_3 = \frac{1}{(1 - M^2 \Lambda^{-2})^4} \left( -\ln(M^2 \Lambda^{-2}) - 3(1 - M^2 \Lambda^{-2}) + \frac{3(1 - M^4 \Lambda^{-4})}{2} - \frac{1 - M^6 \Lambda^{-6}}{3} \right),$$

$$I_4 = \frac{1}{(1 - M^2 \Lambda^{-2})^4} \left( (M^{-2} \Lambda^2 - 1) + 3\ln(M^2 \Lambda^{-2}) + 3(1 - M^2 \Lambda^{-2}) - \frac{(1 - M^4 \Lambda^{-4})}{2} \right),$$

$$I_5 = \frac{1}{(1 - M^2 \Lambda^{-2})^5} \left( (M^{-2} \Lambda^2 - 1) + 4\ln(M^2 \Lambda^{-2}) + 6(1 - M^2 \Lambda^{-2}) - 2(1 - M^4 \Lambda^{-4}) + \frac{1 - M^6 \Lambda^{-6}}{3} \right),$$

$$I_6 = \frac{1}{(1 - M^2 \Lambda^{-2})^5} \left( \frac{M^{-4} \Lambda^4 - 1}{2} - 4(M^{-2} \Lambda^2 - 1) - 6\ln(M^2 \Lambda^{-2}) - 4(1 - M^2 \Lambda^{-2}) + \frac{1 - M^4 \Lambda^{-4}}{2} \right),$$

$$I_7 = \frac{1}{(1 - M^2 \Lambda^{-2})^6} \left( \frac{M^{-4} \Lambda^4 - 1}{2} - 5(M^{-2} \Lambda^2 - 1) - 10\ln(M^2 \Lambda^{-2}) - 10(1 - M^2 \Lambda^{-2}) + \frac{5(1 - M^4 \Lambda^{-4})}{2} - \frac{1 - M^6 \Lambda^{-6}}{3} \right).$$

Note that when  $M \uparrow \Lambda$ ,  $I_1 \rightarrow 1/2$ ,  $I_2 \rightarrow 1/3$ ,  $I_3 \rightarrow 1/4$ ,  $I_4 \rightarrow 1/4$ ,  $I_5 \rightarrow 1/5$ ,  $I_6 \rightarrow 1/5$ ,  $I_7 \rightarrow 1/6$ , implying that  $I_1, \dots, I_7$  are continuous functions of  $M$  when  $M = \Lambda$ . We thus get

$$\text{When } M = \Lambda : \quad F_2(0) = (f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{m}{\Lambda} \cdot \frac{7}{45} + \text{lower order.}$$

We show some of the steps of the calculation of the Feynman integral in detail, for the contributing graph, in Appendix A.

#### IV. CONSTRAINT ON $|f_1 + f_2 + f''|$

Since the experimental value of the anomalous muon magnetic moment exceeds the SM prediction, we can use the experiment minus theory gap as an upper bound on the QED one-loop contribution due to presence of left right symmetric excited muons. Thus, we get that

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} \geq \frac{\alpha}{2\pi} (f_1 + f_2 + f'')^2 \cdot 8Mm\Lambda^{-2} \left( \frac{1}{2}I_1 + \left( 3M^2\Lambda^{-2} - \frac{1}{3} \right) I_2 - 3M^2\Lambda^{-2}I_3 - \frac{3M^4\Lambda^{-4}}{2}I_4 + \frac{3M^4\Lambda^{-4}}{2}I_5 + \frac{M^6\Lambda^{-6}}{3}I_6 - \frac{M^6\Lambda^{-6}}{3}I_7 - \frac{11}{36} \right),$$

where the terms  $I_1, \dots, I_7$  are functions of  $M/\Lambda$  and have been defined earlier. Recall that the assumption behind the above expression on the right hand side was that  $m \ll M \leq \Lambda$ .

There are two current ‘best’ determinations of the experiment minus theory gap for anomalous muon magnetic moment [10] viz.

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 2.37 \times 10^9 \quad \text{OR} \quad a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 2.74 \times 10^9.$$

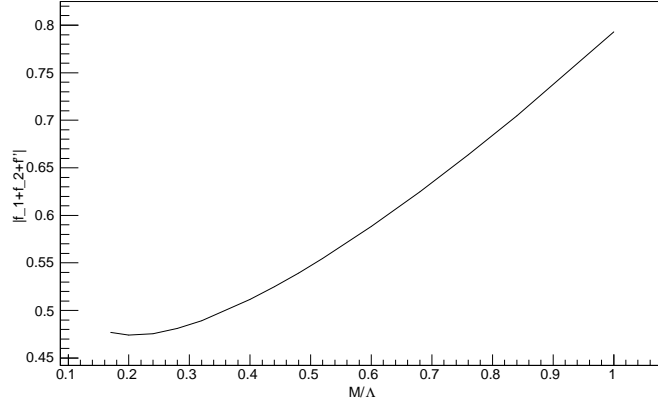


FIG. 2. At  $\Lambda = 5000 \text{ GeV}/c^2$ , for a given value of the ratio  $M/\Lambda$ , the absolute value of the sum of the couplings  $|f_1 + f_2 + f''|$  has to be below the graph shown.

In order to be conservative, we will use the first value for the gap in the calculation above. We will also use the most precise value of  $\alpha$  viz.  $\alpha^{-1} = 137.035999037$  [26], and the most precise value of muon mass viz.  $m = 0.1056583715 \text{ GeV}/c^2$  [4] known currently. Doing so gives us,

$$6.11518 \times 10^{-8} \geq (f_1 + f_2 + f'')^2 \cdot M\Lambda^{-2} \left( \frac{1}{2}I_1 + \left( 3M^2\Lambda^{-2} - \frac{1}{3} \right) I_2 - 3M^2\Lambda^{-2}I_3 - \frac{3M^4\Lambda^{-4}}{2}I_4 \right. \\ \left. + \frac{3M^4\Lambda^{-4}}{2}I_5 + \frac{M^6\Lambda^{-6}}{3}I_6 - \frac{M^6\Lambda^{-6}}{3}I_7 - \frac{11}{36} \right),$$

For  $\Lambda = 5000 \text{ GeV}/c^2$ , the resulting constraint on the absolute value of the sum of the couplings  $|f_1 + f_2 + f''|$  has been plotted in Fig. 2 as a function of  $M/\Lambda$  and found to be modest, of order 1.

## V. CONCLUSION

In this note, we have calculated in detail the QED contribution to anomalous magnetic moment for muons at one loop level arising from left right symmetric excited muons which are the first excited states of known standard model muons. As a result of the gauge invariance of the underlying theory, we have shown that at the one loop level only one Feynman diagram contributes, simplifying our calculations. We obtain constraints on the parameter space of the left right symmetric composite model of [25], which are similar in spirit to the constraints obtained earlier by [13] for left handed excited muons.

Diagram	$iM^\mu$
1.	$-\frac{ee_{\text{eff}}^2}{4} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)(\gamma^\nu \not{k} - \not{k} \gamma^\nu)(q_2 + \not{k} + M)\gamma^\mu(q_1 + \not{k} + M)(\gamma_\nu \not{k} - \not{k} \gamma_\nu)u(q_1)}{(k^2 + i\epsilon)((k+q_1)^2 - M^2 + i\epsilon)((k+q_2)^2 - M^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^4}$
2.	$-\frac{ee_{\text{eff}}^2}{4(p^2 \Lambda^{-2} - 1)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)\gamma^\nu(q_2 + \not{k} + M)(\gamma^\mu \not{p} - \not{p} \gamma^\mu)(q_1 + \not{k} + M)(\gamma_\nu \not{k} - \not{k} \gamma_\nu)u(q_1)}{(k^2 + i\epsilon)((k+q_1)^2 - M^2 + i\epsilon)((k+q_2)^2 - M^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^2}$
2'.	$-\frac{ee_{\text{eff}}^2}{4(p^2 \Lambda^{-2} - 1)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)(\gamma^\nu \not{k} - \not{k} \gamma^\nu)(q_2 + \not{k} + M)(\gamma^\mu \not{p} - \not{p} \gamma^\mu)(q_1 + \not{k} + M)\gamma_\nu u(q_1)}{(k^2 + i\epsilon)((k+q_1)^2 - M^2 + i\epsilon)((k+q_2)^2 - M^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^2}$
3.	$-\frac{ee_{\text{eff}}^2}{4(m^2 - M^2)(p^2 \Lambda^{-2} - 1)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)(\gamma^\mu \not{k} - \not{k} \gamma^\mu)(q_2 - \not{k} + M)\gamma_\nu(q_2 + M)(\gamma^\mu \not{p} - \not{p} \gamma^\mu)u(q_1)}{((k - q_2)^2 - M^2 + i\epsilon)(k^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^2}$
3'.	$-\frac{ee_{\text{eff}}^2}{4(m^2 - M^2)(p^2 \Lambda^{-2} - 1)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)(\gamma^\mu \not{p} - \not{p} \gamma^\mu)(q_1 + M)\gamma^\nu(q_1 - \not{k} + M)(\gamma_\nu \not{k} - \not{k} \gamma_\nu)u(q_1)}{((k - q_1)^2 - M^2 + i\epsilon)(k^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^2}$
4.	$-\frac{ee_{\text{eff}}^2}{4(m^2 - M^2)(p^2 \Lambda^{-2} - 1)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)\gamma^\nu(q_2 - \not{k} + M)(\gamma_\nu \not{k} - \not{k} \gamma_\nu)(q_2 + M)(\gamma^\mu \not{p} - \not{p} \gamma^\mu)u(q_1)}{((k - q_2)^2 - M^2 + i\epsilon)(k^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^2}$
4'.	$-\frac{ee_{\text{eff}}^2}{4(m^2 - M^2)(p^2 \Lambda^{-2} - 1)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(q_2)(\gamma^\mu \not{p} - \not{p} \gamma^\mu)(q_1 + M)(\gamma^\nu \not{k} - \not{k} \gamma^\nu)(q_1 - \not{k} + M)\gamma_\nu u(q_1)}{((k - q_1)^2 - M^2 + i\epsilon)(k^2 + i\epsilon)(k^2 \Lambda^{-2} - 1)^2}$

TABLE I. Feynman graphs and their matrix element contribution to muon magnetic moment.

	Diagram	F <sub>2</sub> (0)
1.		$(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} \cdot I_1(m, M, \Lambda) + \text{low. ord.}$
	$I_1(m, M, \Lambda) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx dy dz \frac{(1-y-z)(1-x-y-z)^3}{[-(y+z)(1-y-z)m^2\Lambda^{-2} + (y+z)M^2\Lambda^{-2} + (1-x-y-z)]^4}.$	

TABLE II. Feynman graph 1 and its contribution to the form factor  $F_2(0)$ .

	Diagram	F <sub>2</sub> (0)
2.		$-(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} \cdot I_2(m, M, \Lambda) - \text{low. ord.}$
2'.		$(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} \cdot I_2(m, M, \Lambda) + \text{low. ord.}$
	$I_2(m, M, \Lambda) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx dy dz \frac{1-x-y-z}{[-(y+z)(1-y-z)m^2\Lambda^{-2} + (y+z)M^2\Lambda^{-2} + (1-x-y-z)]^2}.$	

TABLE III. Feynman graphs 2, 2' and their contribution to the form factor  $F_2(0)$ .

## Appendix A: Details of the calculation for Diagram 1

We now provide some details of the calculation of  $F_2(0)$  for Diagram 1. The Feynman integral is

$$\begin{aligned}
& iM^\mu \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\alpha}}{k^2 + i\epsilon} \\
&\quad \bar{u}(q_2)(e_{\text{eff}}k_\beta\sigma^{\nu\beta}) \frac{1}{(1 - \frac{k^2}{\Lambda^2})^2} \frac{i(q_2 + k + M)}{(q_2 + k)^2 - M^2 + i\epsilon} (-ie\gamma^\mu) \\
&\quad \frac{i(q_1 + k + M)}{(q_1 + k)^2 - M^2 + i\epsilon} (e_{\text{eff}}k_\beta\sigma^{\alpha\beta}) \frac{1}{(1 - \frac{k^2}{\Lambda^2})^2} u(q_1) \\
&= -\frac{ee_{\text{eff}}^2}{4} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(q_2)(\gamma^\nu k - k\gamma^\nu)(q_2 + k + M)\gamma^\mu(q_1 + k + M)(\gamma_\nu k - k\gamma_\nu)u(q_1)}{A_1 A_2 A_3 A_4' A_5' A_6' A_7'},
\end{aligned}$$



	Diagram	$F_2(0)$
3.		$(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{18m}{M-m} \cdot I_3(m, M, \Lambda)$
3'.		$-(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{18m}{M-m} \cdot I_3(m, M, \Lambda)$
$I_3(m, M, \Lambda) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx dy dz \left( \frac{x(1-x)m^2 - xMm}{6\Lambda^2(xM^2\Lambda^{-2} - m^2\Lambda^{-2}x(1-x) + (1-x-y))^2} + \frac{1}{3(xM^2\Lambda^{-2} - m^2\Lambda^{-2}x(1-x) + (1-x-y))} \right)$		

TABLE IV. Feynman graphs 3, 3' and their contribution to the form factor  $F_2(0)$ .

	Diagram	$F_2(0)$
4.		$(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{18m}{M-m} \cdot I_4(m, \Lambda)$
4'.		$-(f_1 + f_2 + f'')^2 \cdot \frac{\alpha}{2\pi} \cdot \frac{18m}{M-m} \cdot I_4(m, \Lambda)$
$I_4(m, \Lambda) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx dy dz \left( \frac{m^2 x^2}{6\Lambda^2(x^2 m^2 \Lambda^{-2} + (1-x-y))^2} - \frac{1}{3(x^2 m^2 \Lambda^{-2} + (1-x-y))} \right)$		

TABLE V. Feynman graphs 4, 4' and their contribution to the form factor  $F_2(0)$ .

where

$$\begin{aligned}
A_1 &= k^2 + i\epsilon, \\
A_2 &= (k + q_1)^2 - M^2 + i\epsilon, \\
A_3 &= (k + q_2)^2 - M^2 + i\epsilon,
\end{aligned}$$

$$A_4 = k^2 - \Lambda^2, \quad A'_4 = \Lambda^{-2} A_4,$$

$$A_5 = A_6 = A_7 = A_4, \quad A'_5 = A'_6 = A'_7 = A'_4.$$

Using standard results about contractions with Dirac matrices, we can simplify the integral to get

$$iM^\mu = -\frac{ee_{\text{eff}}^2 \Lambda^8}{4} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{A_1 A_2 A_3 A_4 A_5 A_6 A_7} \frac{1}{\bar{u}(q_2) [4\cancel{k}(q_1 + \cancel{k})\gamma^\mu(q_2 + \cancel{k})\cancel{k} - 8M(q_1^\mu + k^\mu)\cancel{k}\cancel{k} - 8M(q_2^\mu + k^\mu)\cancel{k}\cancel{k} + 8M^2 k^\mu \cancel{k} + 4\cancel{k}(q_2 + \cancel{k})\cancel{k}(q_1 + \cancel{k})\gamma^\mu + 4\cancel{k}\gamma^\mu(q_1 + \cancel{k})\cancel{k}(q_2 + \cancel{k}) - 4M\cancel{k}\cancel{k}(q_1 + \cancel{k})\gamma^\mu - 4M\cancel{k}\cancel{k}\gamma^\mu(q_2 + \cancel{k}) + 4M^2 \cancel{k}\gamma^\mu \cancel{k}] u(q_1)}.$$

In order to evaluate the above expression, we will complete the square in the denominator. Define  $D = xA_1 + yA_2 + zA_3 + w_4A_4 + w_5A_5 + w_6A_6 + w_7A_7$ , where  $x, y, z, w_4, \dots, w_7 \geq 0$ ,  $x + y + z + w_4 + \dots + w_7 = 1$ . Completing the square using standard techniques gives us

$$D = (k + (y + z)q_1 + zp)^2 - \Delta + (x + y + z)i\epsilon,$$

where  $\Delta = -(y + z)(1 - y - z)m^2 - yzp^2 + (y + z)M^2 + (1 - x - y - z)\Lambda^2$ . This leads to the following expression for  $iM^\mu$ .

$$iM^\mu = -\frac{6!ee_{\text{eff}}^2 \Lambda^8}{4} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx dy dz dw_4 \dots dw_7 \delta(x + y + z + w_4 + \dots + w_7 - 1) \frac{1}{[(k + (y + z)q_1 + zp)^2 - \Delta + (x + y + z)i\epsilon]^7} \frac{1}{\bar{u}(q_2) [4\cancel{k}(q_1 + \cancel{k})\gamma^\mu(q_2 + \cancel{k})\cancel{k} - 8M(q_1^\mu + k^\mu)\cancel{k}\cancel{k} - 8M(q_2^\mu + k^\mu)\cancel{k}\cancel{k} + 8M^2 k^\mu \cancel{k} + 4\cancel{k}(q_2 + \cancel{k})\cancel{k}(q_1 + \cancel{k})\gamma^\mu + 4\cancel{k}\gamma^\mu(q_1 + \cancel{k})\cancel{k}(q_2 + \cancel{k}) - 4M\cancel{k}\cancel{k}(q_1 + \cancel{k})\gamma^\mu - 4M\cancel{k}\cancel{k}\gamma^\mu(q_2 + \cancel{k}) + 4M^2 \cancel{k}\gamma^\mu \cancel{k}] u(q_1)}.$$

Performing the change of variable  $k^\mu \mapsto k^\mu - (y + z)q_1^\mu - zp^\mu = k^\mu - yq_1^\mu - zq_2^\mu$  leads to

$$iM^\mu = -\frac{6!ee_{\text{eff}}^2 \Lambda^8}{4} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx dy dz dw_4 \dots dw_7 \delta(x + y + z + w_4 + \dots + w_7 - 1)$$

$$\frac{N'^\mu}{[k^2 - \Delta + (x + y + z)i\epsilon]^7},$$

where

$$\begin{aligned} N'^\mu = & \bar{u}(q_2)[4(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} + (1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu(\not{k} - y\not{q}_1 + (1 - z)\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2) \\ & - 8M(k^\mu + (1 - y)q_1^\mu - zq_2^\mu)(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2) \\ & - 8M(k^\mu - yq_1^\mu + (1 - z)q_2^\mu)(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2) \\ & + 8M^2(k^\mu - yq_1^\mu - zq_2^\mu)(\not{k} - y\not{q}_1 - z\not{q}_2) \\ & + 4(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 + (1 - z)\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} + (1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \\ & + 4(\not{k} - y\not{q}_1 - z\not{q}_2)\gamma^\mu(\not{k} + (1 - y)\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 + (1 - z)\not{q}_2) \\ & - 4M(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} + (1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \\ & - 4M(\not{k} - y\not{q}_1 - z\not{q}_2)(\not{k} - y\not{q}_1 - z\not{q}_2)\gamma^\mu(\not{k} - y\not{q}_1 + (1 - z)\not{q}_2) \\ & + 4M^2(\not{k} - y\not{q}_1 - z\not{q}_2)\gamma^\mu(\not{k} - y\not{q}_1 - z\not{q}_2)]u(q_1). \end{aligned}$$

Using standard techniques of evaluating Feynman integrals and a fair amount of simplification, it suffices to consider the integral of an expression with the following as numerator instead of  $N'^\mu$ :

$$\begin{aligned} N^\mu = & \bar{u}(q_2)[4(k^2)^2\gamma^\mu \quad I \\ & + 4k^2\gamma^\mu(-y\not{q}_1 + (1 - z)\not{q}_2)(-y\not{q}_1 - z\not{q}_2) \quad II \\ & + 4k^2((1 - y)q_1^\mu - zq_2^\mu)(-y\not{q}_1 - z\not{q}_2) \quad II \\ & - 2k^2(-y\not{q}_1 + (1 - z)\not{q}_2)\gamma^\mu((1 - y)\not{q}_1 - z\not{q}_2) \quad II \\ & - 2k^2(-y\not{q}_1 - z\not{q}_2)\gamma^\mu(-y\not{q}_1 - z\not{q}_2) \quad II \\ & + 4k^2(-yq_1^\mu + (1 - z)q_2^\mu)(-y\not{q}_1 - z\not{q}_2) \quad II \\ & + 4k^2(-y\not{q}_1 - z\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \\ & + 4(-y\not{q}_1 - z\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu(-y\not{q}_1 + (1 - z)\not{q}_2)(-y\not{q}_1 - z\not{q}_2) \quad II \\ & + 4(k^2)^2\gamma^\mu \quad I \\ & + 4k^2(-y\not{q}_1 - z\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \\ & - 2k^2(-y\not{q}_1 + (1 - z)\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \\ & + 4k^2((-yq_1 + (1 - z)q_2) \cdot (-yq_1 - zq_2))\gamma^\mu \quad II \\ & + 4k^2(-y\not{q}_1 - z\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \\ & - 2k^2(-y\not{q}_1 - z\not{q}_2)(-y\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \\ & + 4k^2(-y\not{q}_1 - z\not{q}_2)(-y\not{q}_1 + (1 - z)\not{q}_2)\gamma^\mu \quad II \\ & + 4(-y\not{q}_1 - z\not{q}_2)(-y\not{q}_1 + (1 - z)\not{q}_2)(-y\not{q}_1 - z\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \\ & - 2(k^2)^2\gamma^\mu \quad I \\ & - 2k^2\gamma^\mu(-y\not{q}_1 - z\not{q}_2)(-y\not{q}_1 + (1 - z)\not{q}_2) \quad II \\ & + 4k^2((1 - y)q_1^\mu - zq_2^\mu)(-y\not{q}_1 + (1 - z)\not{q}_2) \quad II \\ & - 2k^2(-y\not{q}_1 - z\not{q}_2)((1 - y)\not{q}_1 - z\not{q}_2)\gamma^\mu \quad II \end{aligned}$$

$$\begin{aligned}
& + 4k^2(-yq_1 - zq_2)\gamma^\mu(-yq_1 + (1-z)q_2) \quad II \\
& - 2k^2(-yq_1 - zq_2)\gamma^\mu(-yq_1 - zq_2) \quad II \\
& + 4k^2(-yq_1 - zq_2)\gamma^\mu((1-y)q_1 - zq_2) \quad II \\
& + 4(-yq_1 - zq_2)\gamma^\mu((1-y)q_1 - zq_2)(-yq_1 - zq_2)(-yq_1 + (1-z)q_2) \quad II \\
& - 2Mk^2\gamma^\mu(-yq_1 - zq_2) \quad III \\
& - 2Mk^2(-yq_1 - zq_2)\gamma^\mu \quad III \\
& - 8Mk^2((1-2y)q_1^\mu + (1-2z)q_2^\mu) \quad III \\
& - 8M((1-2y)q_1^\mu + (1-2z)q_2^\mu)(-yq_1 - zq_2)(-yq_1 - zq_2) \quad II \\
& + 2M^2k^2\gamma^\mu \quad I \\
& + 8M^2(-yq_1^\mu - zq_2^\mu)(-yq_1 - zq_2) \quad II \\
& - 4Mk^2((1-y)q_1 - zq_2)\gamma^\mu \quad III \\
& + 2Mk^2(-yq_1 - zq_2)\gamma^\mu \quad III \\
& - 4Mk^2(-yq_1 - zq_2)\gamma^\mu \quad III \\
& - 4M(-yq_1 - zq_2)(-yq_1 - zq_2)((1-y)q_1 - zq_2)\gamma^\mu \quad II \\
& - 4Mk^2\gamma^\mu(-yq_1 + (1-z)q_2) \quad III \\
& - 4Mk^2(-yq_1^\mu - zq_2^\mu) \quad III \\
& + 2Mk^2(-yq_1 - zq_2)\gamma^\mu \quad III \\
& - 4M(-yq_1 - zq_2)(-yq_1 - zq_2)\gamma^\mu(-yq_1 + (1-z)q_2) \quad II \\
& - 2M^2k^2\gamma^\mu \quad I \\
& + 4M^2(-yq_1 - zq_2)\gamma^\mu(-yq_1 - zq_2) \quad II]u(q_1).
\end{aligned}$$

The terms above marked I are all proportional to  $\gamma^\mu$  and do not contribute to the anomalous magnetic moment. The terms marked II do contribute, but proportional to  $k^2$  or  $M^2$  or  $M$  or 1 in the numerator. Since the denominator behaves like  $(k^2 - \Delta)^7$ , integration over the four momentum  $k$  will give contributions proportional to  $\Delta^{-4}$ ,  $M^2\Delta^{-5}$ ,  $M\Delta^{-5}$  or  $\Delta^{-5}$  respectively. Since  $\Delta$  behaves like  $\Lambda^2$  times a bounded function of  $(x, y, z)$ , the contributions will be of the order of  $\Lambda^{-8}$ ,  $M^2\Lambda^{-10}$ ,  $M\Lambda^{-10}$  and  $\Lambda^{-10}$  respectively. The terms marked III contribute to the anomalous magnetic moment proportional to  $Mk^2$  in the numerator which, after integration over  $k$ , end up being of the order of  $M\Lambda^{-8}$ . Thus, in the regime  $m \ll M \leq \Lambda$  and large  $M$ , the most significant contribution comes from factors of order  $Mk^2$  multiplying  $q_2^\mu$ ,  $q_1^\mu$ . Collecting the terms marked III we get

$$\begin{aligned}
\hat{N}'^\mu &= \bar{u}(q_2)[-2Mk^2\gamma^\mu(-yq_1 - zq_2) - 2Mk^2(-yq_1 - zq_2)\gamma^\mu - 8Mk^2((1-2y)q_1^\mu + (1-2z)q_2^\mu) \\
&\quad - 4Mk^2((1-y)q_1 - zq_2)\gamma^\mu - 4Mk^2\gamma^\mu(-yq_1 + (1-z)q_2) - 4Mk^2(-yq_1^\mu - zq_2^\mu)]u(q_1) \\
&= \bar{u}(q_2)[-4Mk^2(-yq_1^\mu - zq_2^\mu) - 8Mk^2((1-2y)q_1^\mu + (1-2z)q_2^\mu) \\
&\quad + 4Mk^2(1-y)\gamma^\mu q_1 - 8Mk^2(1-y)q_1^\mu + 4Mk^2zq_2\gamma^\mu + 4Mk^2y\gamma^\mu q_1 \\
&\quad + 4Mk^2(1-z)q_2\gamma^\mu - 8Mk^2(1-z)q_2^\mu - 4Mk^2(-yq_1^\mu - zq_2^\mu)]u(q_1) \\
&= \bar{u}(q_2)[-4Mk^2((4-8y)q_1^\mu + (4-8z)q_2^\mu) + 4Mmk^2(1-y)\gamma^\mu + 4Mmk^2z\gamma^\mu
\end{aligned}$$

$$\begin{aligned}
& + 4Mmk^2 y \gamma^\mu + 4Mmk^2 (1-z) \gamma^\mu] u(q_1) \\
& = \bar{u}(q_2) [-4Mk^2 ((4-8y)q_1^\mu + (4-8z)q_2^\mu) + 8Mmk^2 \gamma^\mu] u(q_1).
\end{aligned}$$

The term above proportional to  $\gamma^\mu$  does not contribute to the anomalous magnetic moment. Thus, the main contribution comes from

$$\hat{N}^\mu = \bar{u}(q_2) [-4Mk^2 ((4-8y)q_1^\mu + (4-8z)q_2^\mu)] u(q_1).$$

We now do the substitutions  $2q_2^\mu = (q_2^\mu + q_1^\mu) + p^\mu$ ,  $2q_1^\mu = (q_2^\mu + q_1^\mu) - p^\mu$  and get

$$\hat{N}^\mu = \bar{u}(q_2) [-16Mk^2 ((1-y-z)(q_2^\mu + q_1^\mu) + (y-z)p^\mu)] u(q_1).$$

Applying the Gordon identity, we can simplify the above to get

$$\hat{N}^\mu = \bar{u}(q_2) [-16Mk^2 ((1-y-z)(2m\gamma^\mu - i\sigma^{\mu\nu}p_\nu) + (y-z)p^\mu)] u(q_1).$$

We can check that the  $p^\mu$  term above goes to zero after integrating over  $x, y, z, w_4, \dots, w_7$  because the integrand changes sign on swapping  $y$  and  $z$ . This satisfies the sanity check of the Ward identity. This implies that the only contribution to the anomalous magnetic moment comes from the  $16iMk^2(1-y-z)(\sigma^{\mu\nu}p_\nu)$  term above.

Thus, the leading contribution to the anomalous magnet moment is captured by  $F_2(p^2)$  defined as follows:

$$\begin{aligned}
F_2(p^2) = & -\frac{2m \cdot 6! e e_{\text{eff}}^2 \Lambda^8}{4e} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx dy dz dw_4 \cdots dw_7 \delta(x+y+z+w_4+\cdots+w_7-1) \\
& \frac{16iMk^2(1-y-z)}{[k^2 - \Delta + (x+y+z)i\epsilon]^7} + \text{lower order},
\end{aligned}$$

where  $\Delta = -(1-y-z)(y+z)m^2 - yzp^2 + (1-x-y-z)\Lambda^2 + (y+z)M^2$ . Using standard integration identities for Feynman integrals, we get

$$\begin{aligned}
F_2(p^2) & = -(8iMm \cdot 6! e^2 (f_1 + f_2 + f'')^2 \Lambda^6) \int_0^1 dx dy dz dw_4 \cdots dw_7 \delta(x+y+z+w_4+\cdots+w_7-1) (1-y-z) \\
& \quad \int \frac{d^4 k}{(2\pi)^4} \frac{k^2}{[k^2 - \Delta + (x+y+z)i\epsilon]^7} + \text{lower order} \\
& = \frac{24\alpha}{\pi} \cdot (f_1 + f_2 + f'')^2 \frac{Mm}{\Lambda^2} \int_0^1 dx dy dz dw_4 \cdots dw_7 \delta(x+y+z+w_4+\cdots+w_7-1) \\
& \quad \frac{1-y-z}{[-(1-y-z)(y+z)m^2\Lambda^{-2} - yzp^2\Lambda^{-2} + (1-x-y-z) + (y+z)M^2\Lambda^{-2}]^4} \\
& + \text{lower order}.
\end{aligned}$$

We only need to evaluate  $F_2(0)$ . This gives us the further simplification that

$$\begin{aligned}
F_2(0) = & \frac{24\alpha}{\pi} \cdot \frac{Mm}{\Lambda^2} (f_1 + f_2 + f'')^2 \int_0^1 dx dy dz dw_4 \cdots dw_7 \delta(x+y+z+w_4+\cdots+w_7-1) \\
& \frac{1-y-z}{[(1-x-y-z) + (y+z)M^2\Lambda^{-2} - (1-y-z)(y+z)m^2\Lambda^{-2}]^4}
\end{aligned}$$

+ lower order.

Using a standard integration identity, we get

$$\begin{aligned}
F_2(0) &= \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} (f_1 + f_2 + f'')^2 \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \\
&\quad \frac{(1-y-z)(1-x-y-z)^3}{[(1-x-y-z) + (y+z)M^2\Lambda^{-2} - (1-y-z)(y+z)m^2\Lambda^{-2}]^4} \\
&\quad + \text{lower order} \\
&= \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} (f_1 + f_2 + f'')^2 I_1(m, M, \Lambda) + \text{lower order},
\end{aligned}$$

where

$$I_1(m, M, \Lambda) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx dy dz \frac{(1-y-z)(1-x-y-z)^3}{[-(y+z)(1-y-z)m^2\Lambda^{-2} + (y+z)M^2\Lambda^{-2} + (1-x-y-z)]^4}.$$

We need to evaluate  $F_2(0)$  in the regime  $m \ll M \leq \Lambda$ . Since the integrand is a function of  $y+z$ , we can set  $t = y+z$  and simplify, using the fact that the Jacobian of the transformation  $(y, z)$  maps to  $(y, t)$  is 1, to get

$$\begin{aligned}
F_2(0) &\approx \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} (f_1 + f_2 + f'')^2 \int_0^1 dx \int_0^{1-x} dt \int_0^t dy \frac{(1-t)(1-x-t)^3}{[(1-x-t) + tM^2\Lambda^{-2}]^4} + \text{lower order}.
\end{aligned}$$

After some simplification, this expression becomes

$$\begin{aligned}
F_2(0) &= \frac{\alpha}{2\pi} \cdot \frac{8Mm}{\Lambda^2} (f_1 + f_2 + f'')^2 \int_0^1 dt \\
&\quad \left( \frac{\frac{t}{2} - \frac{t^2}{3}}{1-t+tM^2\Lambda^{-2}} + \frac{3t^2(1-t)M^2\Lambda^{-2}}{1-t+tM^2\Lambda^{-2}} - \frac{3t^3(1-t)M^4\Lambda^{-4}}{2(1-t+tM^2\Lambda^{-2})^2} + \frac{t^4(1-t)M^6\Lambda^{-6}}{3(1-t+tM^2\Lambda^{-2})^3} - \frac{11t(1-t)}{6} \right) \\
&\quad + \text{lower order},
\end{aligned}$$

which has already been stated earlier.

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